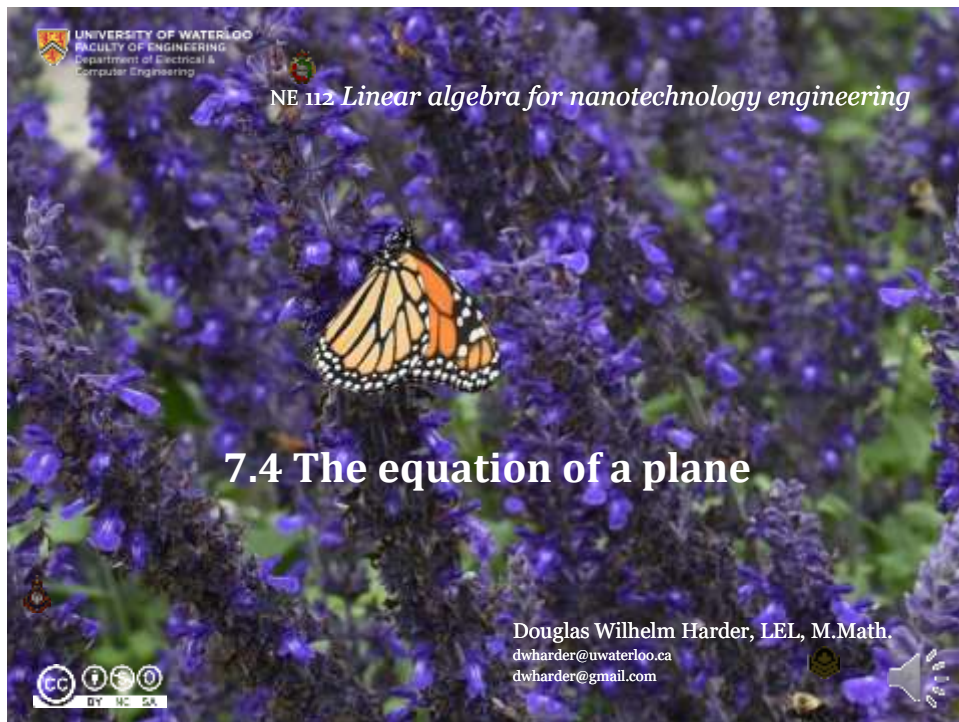
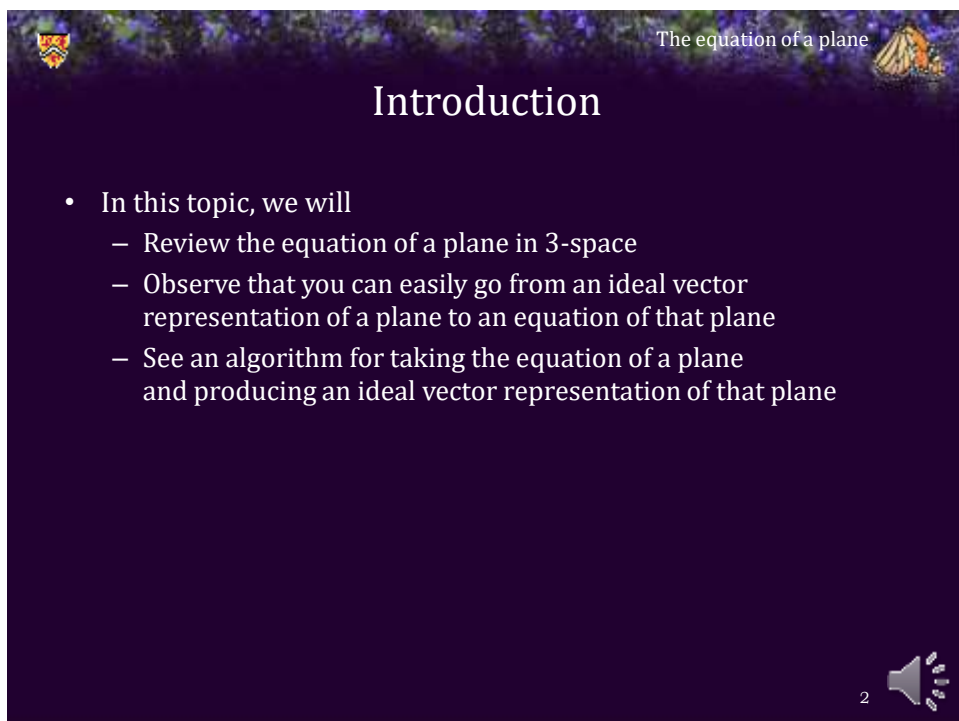




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The equation of a plane

Vector representation of a plane

- In secondary school, you learned that the equation of a plane was

$$ax + by + cz + d = 0$$
- You may recall that the vector $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is orthogonal to the plane
 - Such a vector is *normal* to the plane, hence $\mathbf{n}$
 - Note that given an arbitrary point in 3-space $\mathbf{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, that point is on the plane if $\langle \mathbf{n}, \mathbf{w} \rangle + d = 0$

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The equation of a plane

Vector representation of a plane

- Suppose we are given an ideal vector representation of a plane

$$\mathbf{u}_{\perp} + \alpha \hat{\mathbf{v}}_1 + \beta \hat{\mathbf{v}}_2$$
 - We have that $\mathbf{u}_{\perp}$ is simultaneously a point in the plane and a vector orthogonal to the plane
 - Thus, if we set a, b and c to be the entries of $\mathbf{u}_{\perp}$, it follows that

$$\langle \mathbf{u}_{\perp}, \mathbf{u}_{\perp} \rangle + d = 0$$
 - That is, $d = -\langle \mathbf{u}_{\perp}, \mathbf{u}_{\perp} \rangle$
- With our previous example, $\mathbf{u}_{\perp} = \begin{pmatrix} -3.84 \\ -5.12 \\ 0 \end{pmatrix}$, so our equation of the plane is

$$-3.84x - 5.12y - 40.96 = 0$$
 - We can simplify this to $3x + 4y + 32 = 0$

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The equation of a plane

Vector representation of a plane

- How do you go from the equation of a plane to the vector representation?
 - First, we know that $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is perpendicular to the plane
 - Thus, $\mathbf{u}_\perp = \delta \mathbf{n}$, so it follows that

$$\langle \mathbf{n}, \delta \mathbf{n} \rangle + d = 0$$
 - Therefore,

$$\delta(a^2 + b^2 + c^2) + d = 0$$
 - Thus,

$$\delta = -\frac{d}{a^2 + b^2 + c^2}$$

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The equation of a plane

Vector representation of a plane

- We now have $\mathbf{u}_\perp = \delta \mathbf{n}$, so now we must find two orthonormal vectors that are in the plane
 - We have a preference for a programmatic approach
 - That is, one does not need analysis of intermediate steps, but rather one there is an algorithm that gives us the appropriate vectors
 - Solution: For each of the three canonical basis vectors, find the component perpendicular to normalized normal

$$\mathbf{v}_1 \leftarrow \hat{\mathbf{i}} - \langle \hat{\mathbf{n}}, \hat{\mathbf{i}} \rangle \hat{\mathbf{n}}$$

$$\mathbf{v}_2 \leftarrow \hat{\mathbf{j}} - \langle \hat{\mathbf{n}}, \hat{\mathbf{j}} \rangle \hat{\mathbf{n}}$$

$$\mathbf{v}_3 \leftarrow \hat{\mathbf{k}} - \langle \hat{\mathbf{n}}, \hat{\mathbf{k}} \rangle \hat{\mathbf{n}}$$

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The equation of a plane

Vector representation of a plane

- Now, it may happen that one of the canonical basis vectors parallel or close to parallel to $\mathbf{n}$
 - For this reason, we choose the two vectors that have the maximum 2-norm
 - Relabel these as $\mathbf{v}_1$ and $\mathbf{v}_2$
 - Of these two larger vectors,

it is guaranteed that the 2-norm is no less than $\frac{\sqrt{2}}{2}$
 - It is also guaranteed that the angle between these two vectors is between 60° and 120° , so they will not be parallel to each other

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The equation of a plane

Vector representation of a plane

- Once we have these two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, we just apply Gram-Schmidt to get two orthonormal vectors parallel to the plane

$\hat{\mathbf{v}}_1$

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The equation of a plane

Vector representation of a plane

- For example, find a vector representation of the plane described by equation

$$3x + 4y - 5z + 6 = 0$$
- We start with $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$
- Next, $\delta = -\frac{d}{a^2 + b^2 + c^2} = -\frac{6}{3^2 + 4^2 + 5^2} = -0.12$
- Thus, $\mathbf{u}_\perp = \delta \mathbf{n} = -0.12 \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} = \begin{pmatrix} -0.36 \\ -0.48 \\ 0.6 \end{pmatrix}$

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The equation of a plane

Vector representation of a plane

- Next, we find the perpendicular component of each of the three canonical vectors with respect to $\mathbf{n}$
 - To simplify this, we will use a normalized normal

$$\hat{\mathbf{n}} = \frac{1}{5\sqrt{2}} \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 0.3 \\ 0.4 \\ -0.5 \end{pmatrix}$$

$$\mathbf{v}_1 \leftarrow \hat{\mathbf{i}} - \langle \hat{\mathbf{n}}, \hat{\mathbf{i}} \rangle \hat{\mathbf{n}} = \begin{pmatrix} 0.82 \\ -0.24 \\ 0.3 \end{pmatrix} \quad \mathbf{v}_2 \leftarrow \hat{\mathbf{j}} - \langle \hat{\mathbf{n}}, \hat{\mathbf{j}} \rangle \hat{\mathbf{n}} = \begin{pmatrix} -0.24 \\ 0.68 \\ 0.4 \end{pmatrix} \quad \mathbf{v}_3 \leftarrow \hat{\mathbf{k}} - \langle \hat{\mathbf{n}}, \hat{\mathbf{k}} \rangle \hat{\mathbf{n}} = \begin{pmatrix} 0.3 \\ 0.4 \\ 0.5 \end{pmatrix}$$

- Of these, $\mathbf{v}_1$ has the largest 2-norm, and $\mathbf{v}_2$ has the next largest 2-norm, so by chance, we'll keep these two

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The equation of a plane

Vector representation of a plane

- Apply Gram-Schmidt to these two vectors:

$$\mathbf{v}_1 = \begin{pmatrix} 0.82 \\ -0.24 \\ 0.3 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -0.24 \\ 0.68 \\ 0.4 \end{pmatrix}$$

– Thus,

$$\hat{\mathbf{v}}_1 \leftarrow \begin{pmatrix} 0.9055 \\ -0.2650 \\ 0.3313 \end{pmatrix} \quad \mathbf{v}_2 \leftarrow \mathbf{v}_2 - \langle \hat{\mathbf{v}}_1, \mathbf{v}_2 \rangle \hat{\mathbf{v}}_1 = \begin{pmatrix} 0 \\ 0.6098 \\ 0.4878 \end{pmatrix}$$

$$\hat{\mathbf{v}}_2 \leftarrow \begin{pmatrix} 0 \\ 0.7809 \\ 0.6247 \end{pmatrix}$$

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The equation of a plane

Point on the plane closest to a given point

- Thus, a vector representation of the plane

$$3x + 4y - 5z + 6 = 0$$

is the representation:

$$\mathbf{u}_\perp + \alpha \hat{\mathbf{v}}_1 + \beta \hat{\mathbf{v}}_2 = \begin{pmatrix} -0.36 \\ -0.48 \\ 0.6 \end{pmatrix} + \alpha \begin{pmatrix} 0.9055 \\ -0.2650 \\ 0.3313 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0.7809 \\ 0.6247 \end{pmatrix}$$

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The equation of a plane

Summary

- Following this topic, you now
 - Understand the equation of a plane in 3-space
 - Know that you can easily go from an ideal vector representation of a plane to an equation of that plane
 - Have seen an algorithm for taking the equation of a plane and producing an ideal vector representation of that plane

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The equation of a plane

References

[1] [https://en.wikipedia.org/wiki/Plane_\(geometry\)](https://en.wikipedia.org/wiki/Plane_(geometry))

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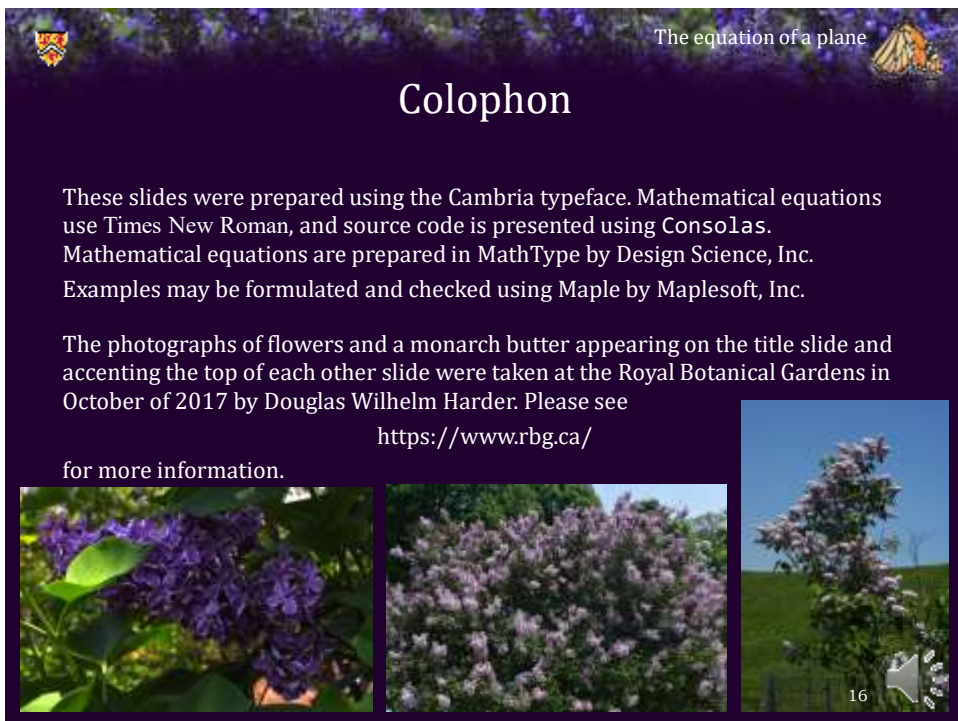
The equation of a plane

Acknowledgments

None so far.

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The equation of a plane

Colophon

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The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see <https://www.rbg.ca/> for more information.



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The equation of a plane

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